Discontinuity surface instead of singularity

M. V. Gorbatenko*

Abstract

Einstein equations are addressed with the energy-momentum tensor that appears if the equations under discussion are required to possess conformal invariance. It is proved that thus derived equations (equations of conformally invariant geometrodynamics) can have not only smooth solutions, but also solutions with discontinuities on space-like hypersurfaces. The solutions obtained are similar to the well-known discontinuous Einstein equation solutions like shock-wave solutions, extended-source solutions, etc.

For the centrally symmetric stationary solution discussed in the paper, the discontinuity surface removes the singularity. The degree of generality of this solution regularization mechanism is discussed.

The issue of the mechanism that forces any smooth solution in the conformally invariant geometrodynamics to be rearranged into the discontinuous one when certain conditions are met is also discussed. The conditions can be: (1) sound speed becoming to be higher than light speed; (2) the solution becoming intolerant to smaller and smaller-scale perturbation modes.

This paper is a summary of ref. [1] that contains missing proofs of the statements presented here and needed references.

We consider discontinuous solutions to Einstein equations with the nonzero energy-momentum tensor. The equations are

$$R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R = T_{\alpha\beta}, \tag{1}$$

where

$$T_{\alpha\beta} = -2A_\alpha A_\beta - g_{\alpha\beta}A^2 - 2g_{\alpha\beta}A^{\nu}_{;\nu} + A_{\alpha;\beta} + A_{\beta;\alpha} + g_{\alpha\beta}\lambda, \tag{2}$$

A_α is a so-called gauge vector, λ is, generally speaking, one of the desired functions. Equations (1) retain their form in conformal transformations

$$g_{\alpha\beta} \to g_{\alpha\beta} \cdot e^{2\sigma}, \quad A_\alpha \to A_\alpha - \sigma_{;\alpha}, \quad \lambda \to \lambda \cdot e^{-2\sigma}. \tag{3}$$

*Russian Federal Nuclear Center - VNIIEF; Sarov, Nizhni Novgorod Region, 607190, Russia; e-mail: gorbatenko@vniief.ru

Therefore we refer to equations (1) with energy-momentum tensor (2) as equations of conformally invariant geometrodynamics.

The Riemannian space complemented with A_α, λ and transformation rules (3)can be considered as Weyl space. In the Weyl space the connectivity $\Gamma^\lambda_{\alpha\beta}$ is expressed in terms of Christoffel symbols $\binom{\lambda}{\alpha\beta}$ and vector A_α with relation

$$\Gamma^\lambda_{\alpha\beta} = \binom{\lambda}{\alpha\beta} + \delta^\lambda_\alpha A_\beta + \delta^\lambda_\beta A_\alpha - g_{\alpha\beta} g^{\lambda\tau} A_\tau. \quad (4)$$

Equations (1) with energy-momentum tensor (2) possess a unique property: Cauchy's problem for these equations is posed without any connections to Cauchy's data. Thus, in the synchronous frame ($g_{0k} = 0$) with using gauge condition $\lambda = Const$ the Cauchy's data are the following sixteen functions: $g_{mn},\ \dot{g}_{mn},\ g_{00},\ A_k$.

The starting axiomatics of the Weyl space requires that the minimum necessary geometric object smoothness classes specified in Table 1 be ensured.

Table 1 - Minimum smoothness classes

Object	Smoothness class as a whole on the manifold	Smoothness class on local charts (pieces)
Components of metrics $g_{\alpha\beta}$	C^1	C^2
Christoffels $\binom{\lambda}{\alpha\beta}$	C^0	C^1
Components of gauge vectorA_α	Piecewise continuous	C^0
Function λ	Piecewise continuous	C^0

In this paper our interest is not with light-like discontinuity surfaces, but with space-like hypersurfaces. Such discontinuity surfaces are known to appear only when the energy-momentum tensor is nonzero. These solutions include gas-dynamic shock-wave type solutions, extended-source solutions, etc. What follows will be analogous to the shock wave theory within the framework of the general relativity theory, but under the sole condition: the energy-momentum tensor will be of form (2).

Apply the requirement of the existence of only minimum smoothness classes to the static centrally symmetric problem. The squared integral for this solution, without loss of generality, can be reduced to

$$ds^2 = -exp\left(\gamma\right) \cdot dt^2 + exp\left(-\gamma\right) \cdot dz^2 + exp\left(\beta\right) \cdot \left(d\theta^2 + sin^2\theta \cdot d\varphi^2\right) \quad (5)$$

and the gauge vector to

$$A_\alpha = \left(\phi, 0, 0, 0\right). \quad (6)$$

For four desired functions $\gamma,\ \beta,\ \phi,\ \lambda$ relations (1) result in four equations which we will not present here. Instead, we at once present the general solution branch, which can transfer to de Sitter solution and Schwartzschield solution.

$$\left.\begin{array}{l} \phi = p \cdot exp\left(\gamma\right) \\ exp\left(\beta\right) = A \cdot sh^2\left(pz + a\right) \\ exp\left(\gamma\right) = \frac{1}{p^2 A} + B \cdot \left[pz \cdot cth\left(pz + a\right) - 1\right] + bp \cdot cth\left(pz + a\right) \\ \lambda\left(z\right) = Bp^2 \end{array}\right\} \quad (7)$$

Seek the solution, such that has a discontinuity for radial variable $z = \Delta$. In region $z < \Delta$ the desired solution is a de Sitter type solution, while in region $z > \Delta$ it is an approximation to the Schwartzschield solution. Denote the constants that describe the desired solution in both the regions as

$$\begin{cases} p_0; \quad A_0; \quad a_0 = 0; \quad B_0; \quad b_0 = 0 \qquad z < \Delta \\ p; \quad A = \frac{1}{p^2}; \quad a; \quad B = 0; \quad b \qquad\qquad z > \Delta \end{cases} \tag{8}$$

The lacing conditions follow from Table 1 and reduce to

$$\left[e^{\beta}\right]\Big|_{\Delta} = \left[e^{\beta}\beta'\right]\Big|_{\Delta} = \left[e^{\gamma}\right]\Big|_{\Delta} = \left[e^{\gamma}\gamma'\right]\big|_{\Delta} = 0, \tag{9}$$

that is to ensuring continuity on the discontinuity surfaces of functions $\gamma, \quad \gamma', \quad \beta, \quad \beta'$. Under assumptions (8) made, conditions (9) determine four of seven problem parameters $p_0;\ A_0;\ B_0;\ \Delta;\ p;\ b;\ a$.

In what follows it would be convenient to introduce parameter $z_0 \equiv -a/p$ in lieu of a and use not parameters $p_0;\ A_0;\ B_0;\ \Delta;\ p;\ b;\ z_0$ themselves, but their dimensionless combinations with retaining the old notations for the dimensionless parameters (excluding z_0),

$$p_0 \to p_0\Delta; \quad p \to p\Delta; \quad A_0 \to \frac{A_0}{\Delta^2}; \quad b \to \frac{b}{\Delta}; \quad \varepsilon \to \frac{z_0}{\Delta}. \tag{10}$$

Upon the transition to the dimensionless parameters the number of the parameters becomes six. Thus, the problem of lacing possesses the self-similarity.

Assume that constants p_0 and $A_0 p_0^2$ are

$$p_0 = 2.77, \quad A_0 p_0^2 = 6.94376 \quad . \tag{11}$$

From conditions (9) we find:

$$p = 2.7887, \quad \varepsilon = 0.954694, \quad B_0 = 2.8803 \cdot 10^{-4}, \quad b = -0.305319. \tag{12}$$

Hence, the accurate solution to the centrally symmetric static problem with the discontinuity surface has been constructed, such that the solution is an analog of the de Sitter solution on the left of the surface and of the Schwartzschield solution on the right.

Energy-momentum tensor (2) can be analyzed by the well-known Eckart method. In gauge

$$\lambda = Const, \quad A^{\mu}_{;\mu} = 0, \tag{13}$$

which is valid in the centrally symmetric static problem as well, the current density vector is written as

$$j_{\alpha} = -2\lambda A_{\alpha}. \tag{14}$$

In the geometrodynamics scheme under discussion, the retained substance density ρ is related to vector j_{α} as

$$j^{\alpha} = \rho \cdot u^{\alpha}. \tag{15}$$

In the centrally symmetric problem,

$$u^{\alpha} = \left(exp\left(-\gamma/2\right);0;0;0\right); \qquad u_{\alpha} = \left(-exp\left(\gamma/2\right);0;0;0\right).$$

The determined energy density U and pressure P result in the following expressions for these quantities:

$$-U \equiv T_0^0 = 3exp\left(-\gamma\right)\cdot\phi^2 + \lambda. \tag{16}$$

$$P \equiv \frac{1}{3}\left(T_1^1 + T_2^2 + T_3^3\right) = exp\left(-\gamma\right)\cdot\phi^2 + \lambda. \tag{17}$$

From (16), (17) it follows that for the static centrally symmetric problem the equation of state of geometrodynamic medium is

$$P = \left(-5/9\right)U. \tag{18}$$

Assume that for the geometrodynamic medium the law of degradation of energy is valid in the form, in which it takes place for viscous heat-conducting continuum. This assumption leads to Maxwell cross relation

$$\frac{\partial}{\partial V}\left(\frac{\partial S}{\partial P}\right) = \frac{\partial}{\partial P}\left(\frac{\partial S}{\partial V}\right). \tag{19}$$

Having solved the differential relation appearing from (19), we arrive at the relation of temperature T to specific volume:

$$T = \frac{Q_0}{V^{1/3}} = Q_0\left(\frac{8}{3}\right)^{1/6}\sqrt{\lambda}. \tag{20}$$

(here Q_0 is some dimensional constant), and entropy density S is

$$S = \frac{3^{2/3}}{2\cdot Q_0\lambda} + S_0. \tag{21}$$

In the stationary centrally symmetric problem under discussion in the region gauge (13) is used in all the thermodynamic quantities are constant and expressed in terms of λ.

The isentropic sound speed for the internal part of the solution is

$$c_s^2 = \frac{7}{3\sqrt{6\lambda}}. \tag{22}$$

From the condition of equal sound speed and light speed we obtain relation

$$c_s^2 = 8\pi r_0. \tag{23}$$

From (22), (23) it follows that the criterion value λ_{cr} of λ, on whose achievement the discontinuity surface appears, is

$$\lambda_{cr} = \frac{7^2}{2^7\cdot 3^3\cdot\pi^2\cdot r_0^2}. \tag{24}$$

The value of $\lambda = \lambda_{cr}$ is established as a result of some dynamic transient, whose description is beyond the scope of this paper. The transient apparently proceeds until λ in the internal part of the solution rises up to a level, at which the sound speed becomes equal to the light speed.

The λ is related to the parameters (10) as $\lambda = B_0 p_0^2$ and the r_0 is nothing else but $r_0 = -b$. Relation (24) for the solution under discussion is therefore equivalent to

$$B_0 p_0^2 b^2 = \frac{7^2}{2^7 \cdot 3^3 \cdot \pi^2}. \tag{25}$$

Thus, the condition of equal sound speed and light speed leads to the relation between the two independent constants p_0, $A_0 p_0^2$. The dependence of $A_0 p_0^2$ on p_0 following from condition (25) is broken at $p_0 = 2.77$, at higher p_0 equation (25) has no solutions. In the range of the values of p_0, at which equation (25) is solvable, the curve $-b\,(p_0)$ monotonically decreases and reaches its minimum at the boundary value $p_0 = 2.77$. In so doing $A_0 p_0^2 = 6.94376$, $b = -0.305319$. It is these boundary values of the constants p_0, $A_0 p_0^2$ that are used in (11).

The existence of the minimum in $-b\,(p_0)$ means that the solution with constants (11) possesses stability in the following sense. If there were no minimum, the parameter $-b\,(p_0)$ could decrease to zero; as the parameter is equal to the Schwartzschield radius, the vanishing would essentially mean the mass "vanishing".

With gauge (13), the dynamic coefficient of viscosity is

$$\eta = \frac{\rho}{2\lambda}. \tag{26}$$

In viscous medium, the solution evolves according to the dynamic equations until Reynolds number R reaches a critical value R_0 leading to appearance of instability of some motion modes, that is turbulence.

Pose the problem to find size L_0, beginning with which the Reynolds number reaches the R_0. From the definition of the Reynolds number it follows that with the system size L_0, characteristic medium velocity u, and mass m observed from the outside the following relation takes place:

$$L_0 = \frac{R_0 c^3}{u \cdot 16\pi G \lambda m}. \tag{27}$$

To find L_0, as it follows from (27), some assumptions of the value of u as well as an assumption allowing λ to be estimated have to be made.

As for u, assume that it characterizes the perturbation velocity, that is take

$$u = c. \tag{28}$$

In view of this assumption, relation (27) becomes

$$L_0 = \left(\frac{R_0}{16\pi}\right) \cdot \frac{1}{\lambda r_0} \quad , \tag{29}$$

where r_0 is the gravitational radius of the object. Assume that the coefficient in brackets is on the order of one. Then from (29) it follows that

$$L_0 \sim \frac{1}{\lambda r_0}. \tag{30}$$

Find the characteristic size L_0 for the above-constructed discontinuous stationary centrally symmetric geometrodynamic solution. We obtain

$$L_0 \sim \frac{2^7 \cdot 3^3 \cdot \pi^2 \cdot r_0}{7^2} \approx 120 \cdot r_0. \tag{31}$$

Thus, the intolerance of the centrally symmetric solution to perturbations can appear at the values of the radial variable L_0 much higher than the gravitational radius. However, its specific value depends on the choice of the Reynolds number for the criterion value.

Hence, equations (1) with tensor $T_{\alpha\beta}$ of form (2) admit the existence of discontinuous solutions. The possibility in itself to construct the discontinuity-surface solution is a nontrivial fact. Note that, for example, the geometrodynamic equations without λ term do not admit construction of the solution obtained in the paper.

The discontinuous solution constructed is regular in the entire radial variable range, which the geodesic completeness takes place in. What makes the solution to be rearranged with the transition of the Schwartzschield branch to the de Sitter branch at certain radial variable? The question touches upon the fundamental problem of the unlimited energy cumulation instability, that is the problem addressed in many papers. We do not have a full answer to the question, at least, for the reason that in this paper we restrict our consideration to the scope of the static centrally symmetric problem. However, we can specify the peculiar geometrodynamics features that can lead to the singularity removal by the discontinuity surface formation.

First, in the scheme under discussion the space-time evolution proceeds in accordance with the dynamic equations at any time. No manual introduction of equations of state or coefficients of viscosity is admitted. As a result, meeting the energy dominance condition, which collapses are typically associated with, is not guaranteed whatsoever.

Second, with increasing λ viscosity decreases, Reynolds number increases. When the Reynolds number has reached some criterion value R_0, the solution becomes intolerant to the perturbations of characteristic size (31). It is hard to tell as applied to geometrodynamic medium at what specific value of R_0 the perturbations will rearrange the solution. But that the solution rearrangement is inevitable follows from the fact that smaller and smaller-scale perturbations become unstable as the singularity is approached. At some stage the turbulization will come to be caused by quantum fluctuations of vacuum.

The above-mentioned features of the dynamic equations allow us to suggest that the mechanism of the solution regularization in geometrodynamics through the discontinuity surface formation can be of general nature. This paper has validated this suggestion by the specific example.

The work was carried out under support in part by the International Science and Technology Center (Project # KR-677).

References

[1] M.V. Gorbatenko. *Voprosy Atomnoi Nauki i Tekhniki.* Seriya: Teor. i Prikl. Fizika. **1-2**, 9-21 (2002) [In Russian].